International Journal of Theoretical Physics, Vol. 10, No. 1 (1974), pp. 19–37

Quantization of Non-Local Field Theory

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Received: 25 *June* 1973

Abstract

The scheme of quantisation of non-local field theory is formulated.

An intermediate regularisation is introduced into the non-local Lagrangian of the classical scalar field in such a way that the procedure of the canonical quantisation leads to the appearance of additional ghost states with indefinite metrics. The ghost states disappear when the regularisation is removed but the propagator of the scalar particle becomes non-local and the S-matrix is finite, unitary, causal and covariant in each perturbation order.

1. *Introduction*

The self-consistent non-local quantum field theory was formulated by Efimov (1968, 1970) in the framework of the Bogolubov, Mevedev & Polivanov (1958) and Bogolubov & Shirkov (1969) axiomatics.

The outline of the construction of the S-matrix in the non-local theory consisted of the following. The general axioms were formulated, then a certain non-local Lagrangian of the classical field was considered. For example, in the case of the one-component scalar field $\varphi(x)$ the Lagrangian of this kind can be

$$\mathscr{L}(x) = \tfrac{1}{2}\varphi(x)(\Box - m^2)\varphi(x) - g\left\{\int dyK(x-y)\varphi(y)\right\}^4 \tag{1.1}$$

where $K(x-y)$ is a non-local form factor. In order to construct the S-matrix in perturbation theory we used the correspondence principle which states that for infinitesimal g the S-matrix has the form

$$S = 1 - ig\int dx\colon \left\{\int dyK(x-y)\varphi(y)\right\}^4 \tag{1.2}$$

where $\varphi(x)$ is the quantised scalar field which satisfies

$$(\Box - m^2)\varphi(x) = 0 \tag{1.3}$$

In constructing the highest perturbation orders we then introduced a reaction, according to which the propagator of the scalar particle changes into the following:

$$\frac{1}{m^2 - k^2 - i\epsilon} \to \frac{[\tilde{K}(k)]^2}{m^2 - k^2 - i\epsilon} \tag{1.4}$$

We also indicated the class of functions to which the form factor $[\tilde{K}(k)]^2$ belongs and introduced an intermediate regularisation in the framework of which the finite S-matrix was constructed. After the regularisation had been removed we proved that the S-matrix satisfies all general axioms: unitarity, causality, covariance, gauge invariance and so on.

In this sense our scheme is purely constructive because it gives some prescriptions within the bounds of which it is possible to construct a finite S-matrix satisfying all initial axioms. However, the suggested scheme is not connected with any canonical quantisation of non-local classical fields of the type (1.1).

In the present paper we propose a procedure of canonical quantisation of non-local fields which are described by the Lagrangian of the type (1.1). Our idea is as follows: We introduce a certain regularisation into the classical Lagrangian in such a way which permits us to carry out the usual canonical quantisation. This quantisation leads to the appearance of additional ghost states with indefinite metrics. The ghost states disappear when the regularisation is removed, but a trace remains, namely the propagator of the scalar particle becomes non-local according to (1.4).

Distant analogy of the procedure proposed is the method of quantisation of the electromagnetic field (see, for example, Bogolubov & Shirkov (1969). The real physical photon can be in two states with transverse polarisations only. However, these states cannot lead to the correct propagator of the photon. In order to get the propagator of the virtual photon in the form

$$D_{\mu\nu}(k^2) = \frac{\delta_{\mu\nu}}{-k^2 - i\epsilon} \tag{1.5}$$

it is necessary to introduce non-physical scalar and longitudinal quanta with indefinite metrics into consideration. The Lorentz condition

$$\partial_\mu A_\mu^{(-)}(x)\,|\,\text{any physical state}\rangle = 0$$

and the current conservation $\partial_\mu J_\mu(x) = 0$ guarantee that the scalar and longitudinal quanta do not appear in any interactions of physical transverse photons with electrons. However, their role is that they contribute to the propagator (1.5), i.e. the virtual photon consists of transverse, longitudinal and scalar quanta.

2. *Formulation of the Quantisation Problem*

Let us consider a one-component scalar field $\varphi(x)$. The Lagrangian density describing the classical field $\varphi(x)$ can be written in the form

$$\mathscr{L}(x) = \tfrac{1}{2}\varphi(x)(\Box - m^2)\varphi(x) + gU(K(l^2\Box)\varphi(x)) \tag{2.1}$$

Here $\Box = -(\partial^2/\partial x_0^2) + (\partial^2/\partial \mathbf{x}^2)$, l is a given parameter having the dimension of length, g is the coupling constant, $U(w)$ is a function describing self-interaction of the scalar field $\varphi(x)$. We suppose that $U(w)$ is analytical in the neighbourhood of the real axis in the complex $w = u + iv$-plane. In all other respects it is an arbitrary function. The classes of functions $U(w)$ for which the finite non-local S-matrix can be constructed are considered in detail in Alebastrov & Efimov (1972, 1973).

The operator $K(l^2\Box)$in (2.1) is non-local and can be presented in the form

$$K(l^2\Box) = \sum_{n=0}^{\infty} \frac{C_n}{(2n)!}(l^2\Box)^n \tag{2.2}$$

In what follows, for convenience we consider the operators

$$V(l^2\Box) = [K(l^2\Box)]^2 \tag{2.3}$$

and suppose that the function $V(z)$ satisfies the following conditions:

(1) $V(z)$ is an entire function of the finite order $\frac{1}{2} \leqslant \rho < 1$, i.e. $\exists\, C > 0$, $b > 0$

$$|V(z)| \leqslant C \exp(b|z|^\rho),$$

(2) $[V(z)]^* = V(z^*)$,

(3) $V(m^2l^2) = 1$,

(4) $V(x) \geqslant 0$ for real x,

(5)
$$V(z) = \begin{cases} O\left(\dfrac{1}{|z|^2}\right), & \operatorname{Re} z \to -\infty \\ O(\exp\{b|z|^\rho\}), & \operatorname{Re} z \to +\infty \end{cases}$$

In the expansion

(6) $V(z) = \sum_{n=0}^{\infty} v_n[z - m^2l^2]^n$

the coefficients v_n satisfy

(7) $v_0 = 1$, $v_n > 0$, $\forall n$,

(8) $\exists\, C > 0$, $A > 0$ that

$$v_n < C\frac{A^n}{\Gamma\left(\dfrac{n}{\rho} + 1\right)}$$

Thus we have described all the quantities in the Lagrangian density (2.1).

The wave equation for the classical scalar field $\varphi(x)$ can be obtained according to the principle of stationary action and may be represented in the form

$$(\Box - m^2)\varphi(x) = -gK(l^2\Box)U'(K(l^2\Box)\varphi(x)) \tag{2.4}$$

Let us introduce the field

$$\phi(x) = K(l^2\Box)\varphi(x) \tag{2.5}$$

Then the total Lagrangian (2.1) for the field $\phi(x)$ is

$$\mathscr{L}(x) = \tfrac{1}{2}\phi(x)E(\Box)\,\phi(x) + gU(\phi(x)) \tag{2.6}$$

where

$$E(\Box) = \frac{\Box - m^2}{V(l^2\Box)} \tag{2.7}$$

Formally the wave equation is written as

$$E(\Box)\phi(x) = -gU'(\phi(x)) \tag{2.8}$$

for the interacting field, and

$$E(\Box)\phi(x) = \frac{\Box - m^2}{V(l^2\Box)}\,\phi(x) = 0 \tag{2.9}$$

for the free field.

The problem is how to understand these equations, how to investigate and solve them and how to perform the quantisation of the field $\phi(x)$.

We proceed in the following way. Instead of equations (2.8) or (2.9) we consider a regularised equation

$$E^\delta(\Box)\phi^\delta(x) = -gU'(\phi^\delta(x)) \tag{2.10}$$

or, in the case $g = 0$,

$$E^\delta(\Box)\phi^\delta(x) = 0$$

Here δ is a parameter of the regularisation such that

$$\lim_{\delta\to 0} E^\delta(\Box) = E(\Box) = \frac{\Box - m^2}{V(l^2\Box)} \tag{2.11}$$

Accordingly, instead of the Lagrangian (2.6) we obtain

$$\mathscr{L}^\delta(x) = \tfrac{1}{2}\phi^\delta(x)E^\delta(\Box)\phi^\delta(x) + gU(\phi^\delta(x)) \tag{2.12}$$

Our regularisation is chosen in such a way that the function

$$E^\delta(k^2) = \frac{k^2 - m^2}{V^\delta(l^2k^2)}$$

has zeros in some set of points

$$E^{\delta}(k^2) \sim (k^2 - m^2)\prod_{j=1}^{\infty}(k^2 - m_j^2(\delta)) \tag{2.13}$$

We suppose that

$$m_j^2(\delta) > 0 \qquad (j = 1, 2, 3, \ldots)$$

and

$$m_j^2(\delta) \to \infty$$

when $\delta \to 0$.

Then the field $\phi^{\delta}(x)$ $(\delta > 0)$ can be quantised by the methods with indefinite metrics (see papers of Blokhintsev (1947), Pais & Uhlenbeck (1950) and Nagy (1966)).

The Hamiltonian H_0^{δ} and the vector space of states $\mathscr{H}^{\delta}$ with indefinite metrics can be constructed for the free system when $\delta > 0$. Further the S^{δ}-matrix can be found for the interacting system.

By definition, we consider that when $\delta \to 0$ the limits of all physical quantities (S-matrix, current operator $J(x) = i(\delta S/\delta\phi)S^{+}$, Green function $G(x_1, \ldots, x_n)$ and so on) are the quantum field solution of the initial system (2.6).

The problem is to indicate such a regularisation procedure which provides the existence of the limits of all operators and matrix elements for any physical quantities at $\delta \to 0$. It means that we have to obtain a self-consistent theory in the limit $\delta \to 0$. This paper is devoted to this problem.

3. *Regularisation Procedure*

The regularisation is introduced in the following way. Instead of the function

$$V(k^2 l^2) = \sum_{n=0}^{\infty} v_n l^{2n}(k^2 - m^2)^n \tag{3.1}$$

we introduce the regularised function

$$V^{\delta}(k^2 l^2) = \sum_{n=0}^{\infty} \frac{v_n l^{2n}(k^2 - m^2)^n}{\prod_{j=1}^{n+2}\left(1 - \frac{\delta}{j}\frac{k^2 - m^2}{m^2}\right)} \tag{3.2}$$

It is convenient to denote

$$z = \frac{k^2}{m^2}, \qquad \lambda = m^2 l^2 \tag{3.3}$$

Then

$$V^{\delta}(z\lambda) = \sum_{n=0}^{\infty} \frac{v_n \lambda^n (z-1)^n}{\prod_{j=1}^{n+2} \left(1 - \frac{\delta}{j}(z-1)\right)} \tag{3.4}$$

Let us consider the function

$$E^{\delta}(k^2) = \frac{k^2 - m^2}{V^{\delta}(k^2 l^2)} = m^2 \frac{z-1}{V^{\delta}(z\lambda)} \tag{3.5}$$

The following representation is valid

$$\frac{1}{E^{\delta}(k^2)} = \frac{1}{m^2} \frac{V^{\delta}(z\lambda)}{z-1} = \frac{1}{m^2} \sum_{j=0}^{\infty} \frac{(-)^j A_j^{\delta}}{z - \mu_j} \tag{3.6}$$

Here

$$\mu_j = 1 + \frac{j}{\delta} \qquad (j = 0, 1, 2, \ldots) \tag{3.7}$$

$$A_j^{\delta} = \sum_{n=\max\{0, j-2\}}^{\infty} v_n \lambda^n \frac{(n+2)!}{j!(n+2-j)!} \left(\frac{j}{\delta}\right)^n \tag{3.8}$$

$$\sum_{j=0}^{\infty} (-)^j A_j^{\delta} \mu_j^s = 0 \qquad \text{for } s = 0, 1 \tag{3.9}$$

The numbers A_j^{δ} can be represented in the form

$$A_j^{\delta} = \frac{1}{2\pi i} \oint \frac{d\zeta (1+\zeta)^2}{\zeta^{j+1}} V\left((1+\zeta)\frac{j}{\delta}\lambda\right)$$

It is easy to obtain the following estimate

$$A_j^{\delta} \leqslant \min_N \frac{(1+N)^2}{N^j} V\left((1+N)\frac{j}{\delta}\lambda\right) \leqslant C \min_N \frac{(1+N)^2}{N^j} \exp\left\{b\left[(1+N)\frac{j}{\delta}\lambda\right]^{\rho}\right\}$$

$$\gtrsim \text{const.}\, [B(\delta)]^j \exp\left[-\left(\frac{1-\rho}{\rho}\right) j \log j\right] \qquad (j \to \infty) \tag{3.10}$$

The function $D^{\delta}(k^2) = [E^{\delta}(k^2)]^{-1}$ has the properties:

(1) It is a meromorphic analytical function in the complex k^2-plane and has the simple poles at the points

$$m_j^2(\delta) = m^2 \mu_j = m^2 \left(1 + \frac{j}{\delta}\right) \qquad (j = 0, 1, 2, \ldots)$$

(2) The residues of $D^{\delta}(k^2)$ at these points are

$$\operatorname*{Res}_{k^2=m_j^2(\delta)} D^{\delta}(k^2) = (-)^j A_j^{\delta}$$

(3) When $|k^2| \to \infty$ in all k^2-plane, except the ray $[m^2, \infty)$.

$$D^{\delta}(k^2) = \frac{V^{\delta}(k^2l^2)}{k^2 - m^2} = O\left(\frac{1}{|k^2|^3}\right)$$

(4) The function $D^{\delta}(k^2)$ may have zeros at the points a_p ($p = 1, 2, 3, \ldots$)

(5)
$$\lim_{\delta \to 0} D^{\delta}(k^2) = \frac{V(k^2l^2)}{k^2 - m^2}$$

4. *Quantisation of the Regularised Equation*

Let us consider the classical system described by the Lagrangian density:

$$\mathscr{L}^{\delta}(x) = \tfrac{1}{2}\phi^{\delta}(x)E^{\delta}(\Box)\phi^{\delta}(x) + gU(\phi^{\delta}(x)) \tag{4.1}$$

where the regularised operator $E^{\delta}(\Box)$ satisfies the properties enumerated above.

According to the principle of stationary action, the wave equation for the system described by (4.1) has the form

$$E^{\delta}(\Box)\phi^{\delta}(x) = -gU'(\phi^{\delta}(x)) \tag{4.2}$$

It is the differential equation of the infinite order, i.e. it is an integral equation. In order to solve the Cauchy problem we have to know the values of the function $\phi^{\delta}(x)$ and all its derivatives in the initial moment of time.

We analyse the solving of this equation following the scheme proposed by Pais and Uhlenbeck (1950). Let us introduce a system of fields

$$\phi_j^{\delta}(x) = \sqrt{(A_j^{\delta})}\frac{E^{\delta}(\Box)}{\Box - m_j^2(\delta)}\phi^{\delta}(x) \tag{4.3}$$

where

$$m_j^2(\delta) = m^2\left(1 + \frac{j}{\delta}\right) \qquad (j = 0, 1, 2, \ldots) \tag{4.4}$$

According to definition (4.3), the fields $\phi_j^{\delta}(x)$ are not independent for different j and they satisfy the correlations

$$\sqrt{(A_j^{\delta})}\frac{E^{\delta}(\Box)}{\Box - m_j^2(\delta)}\phi_i^{\delta}(x) = \sqrt{(A_i^{\delta})}\frac{E^{\delta}(\Box)}{\Box - m_i^2(\delta)}\phi_j^{\delta}(x) \tag{4.5}$$

The field $\phi^{\delta}(x)$ can be expressed as

$$\phi^{\delta}(x) = \sum_{j=0}^{\infty}(-)^j\sqrt{(A_j^{\delta})}\phi_j^{\delta}(x) \tag{4.6}$$

In fact, on the one hand, the following chain of equalities

$$\phi^{\delta}(x) = \sum_{j=0}^{\infty} (-)^j \sqrt{(A_j^{\delta})}\sqrt{(A_j^{\delta})}\frac{E^{\delta}(\Box)}{\Box - m_j^2(\delta)}\phi^{\delta}(x)$$

$$= \sum_{j=0}^{\infty} \frac{(-)^j A_j^{\delta}}{\Box - m_j^2(\delta)} E^{\delta}(\Box)\phi^{\delta}(x) = \frac{1}{E^{\delta}(\Box)} E^{\delta}(\Box)\phi^{\delta}(x) = \phi^{\delta}(x)$$

is valid. On the otner hand, using (4.5) it is possible to obtain

$$\phi_j^{\delta}(x) = \sqrt{(A_j^{\delta})}\frac{E^{\delta}(\Box)}{\Box - m_j^2(\delta)}\sum_{i=0}^{\infty} (-)^i \sqrt{(A_i^{\delta})}\phi_i^{\delta}(x)$$

$$= \sum_{i=0}^{\infty} (-)^i \sqrt{(A_i^{\delta})}\sqrt{(A_i^{\delta})}\frac{E^{\delta}(\Box)}{\Box - m_i^2(\delta)}\phi_j^{\delta}(x) = \sum_{i=0}^{\infty} \frac{(-)^i A_i^{\delta}}{\Box - m_i^2(\delta)} E^{\delta}(\Box)\phi_j^{\delta}(x)$$

$$= \phi_j^{\delta}(x)$$

On the basis of the correlations (4.3), (4.5) and (4.6) the Lagrangian density can be expressed in terms of the fields $\phi_j^{\delta}(x)$;

$$\mathscr{L}^{\delta}(x) = \tfrac{1}{2} \sum_{j=0}^{\infty} (-)^j \phi_j^{\delta}(x)(\Box - m_j^2(\delta))\phi_j^{\delta}(x) + gU\left(\sum_{j=0}^{\infty} (-)^j \sqrt{(A_j^{\delta})}\phi_j^{\delta}(x)\right) \tag{4.7}$$

Equation (4.2) can be written in the form of the infinite system of equations

$$(\Box - m_j^2(\delta))\phi_j^{\delta}(x) = -g\sqrt{(A_j^{\delta})}U'\left(\sum_{s=0}^{\infty} (-)^s \sqrt{(A_s^{\delta})}\phi_s^{\delta}(x)\right) \qquad (j = 0, 1, 2, \ldots) \tag{4.8}$$

In fact substituting (4.6) into (4.2) and making use of (4.5) it is possible to obtain

$$E^{\delta}(\Box)\phi^{\delta}(x) = E^{\delta}(\Box)\sum_{j=0}^{\infty} (-)^j \sqrt{(A_j^{\delta})}\phi_j^{\delta}(x)$$

$$= \sum_{j=0}^{\infty} (-)^j \sqrt{(A_j^{\delta})}\frac{1}{\sqrt{(A_k^{\delta})}}(\Box - m_k^2(\delta))\sqrt{(A_k^{\delta})}\frac{E^{\delta}(\Box)}{\Box - m_k^2(\delta)}\phi_j^{\delta}(x)$$

$$= \frac{\Box - m_k^2(\delta)}{\sqrt{(A_k^{\delta})}}\sum_{j=0}^{\infty} \frac{(-)^j A_j^{\delta}}{\Box - m_j^2(\delta)} E^{\delta}(\Box)\phi_k^{\delta}(x)$$

$$= \frac{\Box - m_k^2(\delta)}{\sqrt{(A_k^{\delta})}}\phi_k^{\delta}(x) = -gU'\left(\sum_{s=0}^{\infty} (-)^s \sqrt{(A_s^{\delta})}\phi_s^{\delta}(x)\right)$$

From here it follows (4.8).

Thus the Lagrangian (4.7) and the system of (4.8) are completely equivalent to the Lagrangian (4.1) and equation (4.2).

Proving this equivalence we have considered that the fields $\phi_j^\delta(x)$ are not independent because they are defined by the correlation (4.3). However the representation of the Lagrangian in the form (4.7) and the wave equation in the system (4.8) allows us to consider these fields, $\phi_j^\delta(x)$, as being completely independent.

This method is well known in the theory of differential equations. It is used usually when a differential equation of the highest order is replaced by a system of differential equations of the first order.

Starting from the Lagrangian (4.7) which describes the system of the independent fields $\{\phi_j^\delta\}$ and leads to the system of equations (4.8) it is easy to show that the field

$$\phi^\delta(x) = \sum_{j=0}^{\infty} (-)^j \sqrt{(A_j^\delta)}\phi_j^\delta(x)$$

satisfies equation (4.2), and correlation (4.5) is valid.

Thus we can consider that our initial system (4.1) is described by the Lagrangian (4.7) where the fields are independent and satisfy the wave equations (4.8).

All the above-stated arguments concerned the classical field theory. The quantisation of the system of the classical fields $\{\phi_j^\delta(x)\}$ can be performed according to the canonical procedure of quantisation (see, for example, Wentzel (1943)). Let us introduce a momentum field, conjugate to $\phi_j^\delta(\mathbf{x}, 0)$

$$\Pi_j^\delta(\mathbf{x}, 0) = \frac{\delta}{\delta\dot{\phi}_j^\delta(\mathbf{x}, 0)} \int d\mathbf{y}\mathscr{L}^\delta(\mathbf{y}, 0) = (-)^j\dot{\phi}_j^\delta(\mathbf{x}, 0) \qquad (4.9)$$

We treat ϕ_j^δ and Π_j^δ as operators with the commutation relations:

$$\begin{aligned} &[\phi_j^\delta(\mathbf{x}, 0), \phi_{j'}^\delta(\mathbf{x}', 0)]_- = [\Pi_j^\delta(\mathbf{x}, 0), \Pi_{j'}^\delta(\mathbf{x}', 0)]_- = 0 \\ &[\phi_j^\delta(\mathbf{x}, 0), \Pi_{j'}^\delta(\mathbf{x}', 0)]_- = i\delta_{jj'}\delta(\mathbf{x} - \mathbf{x}') \end{aligned} \qquad (4.10)$$

or

$$[\phi_j^\delta(\mathbf{x}, 0), \dot{\phi}_{j'}^\delta(\mathbf{x}', 0)]_- = i(-)^j\delta_{jj'}\delta(\mathbf{x} - \mathbf{x}') \qquad (4.11)$$

It is seen that the indefinite metrics is to be employed to quantise our system in a regular manner (see Nagy (1966)).

As we are unable to solve the system of equations (4.8) exactly, our problem is to construct the perturbation series for the S-matrix and we perform the quantisation of the non-interacting system of fields $\{\phi_j^\delta(x)\}$. Instead of (4.8) we have

$$(\Box - m_j^2(\delta))\phi_j^\delta(x) = 0 \qquad (j = 0, 1, 2, \ldots) \qquad (4.12)$$

The solution of these equations can be written in the form

$$\phi_j^\delta(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d\mathbf{k}}{\sqrt{(2\omega_{j\mathbf{k}}^\delta)}} (d_{j\mathbf{k}} e^{-ikx} + d_{j\mathbf{k}}^+ e^{ikx}) \tag{4.13}$$

$$\omega_{j\mathbf{k}}^\delta = \sqrt{[\mathbf{k}^2 + m_j^2(\delta)]} = \sqrt{\left[[\mathbf{k}^2 + m^2 \left(1 + \frac{j}{\delta}\right)\right]}^{1/2}$$

The commutation relations (4.10) and (4.11) leads to the operators $d_{j\mathbf{k}}$ and $d_{j\mathbf{k}}$ to satisfy

$$\begin{aligned} &[d_{j\mathbf{k}}, d_{j'\mathbf{k}'}]_- = [d_{j\mathbf{k}}^+, d_{j'\mathbf{k}'}^+]_- = 0 \\ &[d_{j\mathbf{k}}, d_{j'\mathbf{k}'}^+]_- = (-)^j \delta_{jj'} \delta(\mathbf{k} - \mathbf{k}') \end{aligned} \tag{4.14}$$

The Hamiltonian of the non-interacting system can easily be obtained

$$H_0{}^\delta = \sum_{j=0}^{\infty} (-)^j \int d\mathbf{k} \omega_{j\mathbf{k}}^\delta d_{j\mathbf{k}}^+ d_{j\mathbf{k}} \tag{4.15}$$

The system under consideration consists of quanta with the following mass spectra

$$m_j^2(\delta) = \begin{cases} m^2, & j = 0 \\ m^2 \left(1 + \dfrac{j}{\delta}\right), & j = 1, 2, 3, \ldots \end{cases} \tag{4.16}$$

Let us denote

$$d_{0\mathbf{k}} = a_\mathbf{k}, \qquad d_{0\mathbf{k}}^+ = a_\mathbf{k} \tag{4.17}$$

When $\delta \to 0$, the masses of quanta with $j = 1, 2, 3, \ldots$ go to infinity, according to (4.16). These quanta are called ghost states or ghosts. The quanta with $j = 0$ have the finite mass m. We call them normal particles or scalar particles with mass m.

The space of states $\mathscr{H}^\delta$ is a vector space with indefinite metrics. It consists of

(1) a vacuum state $|0\rangle$, that is unique, defined by the conditions

$$d_{j\mathbf{k}}|0\rangle = 0$$

and normalised by $\langle 0|0\rangle = 1$;

(2) one-particle states $|j, \mathbf{k}\rangle = d_{j\mathbf{k}}^+|0\rangle$ which are normalised by

$$\langle j, \mathbf{k}|j', \mathbf{k}'\rangle = (-)^j \delta_{jj'} \delta(\mathbf{k} - \mathbf{k}')$$

and are eigenstates of the Hamiltonian $H_0{}^\delta$

$$H_0{}^\delta |j, \mathbf{k}\rangle = \omega_{j\mathbf{k}}^\delta |j, \mathbf{k}\rangle;$$

(3) many-particle states. If there are n particles with momenta $\mathbf{k}_1, \ldots, \mathbf{k}_n$ and among them there are $\nu_1, \nu_2, \ldots, \nu_\alpha$ $(n = \nu_1 + \nu_2 + \cdots + \nu_\alpha)$ identical particles, i.e. with the same index j, then the following is given:

$$|n\rangle = |j_1, \mathbf{k}_1; \ldots; j_n, \mathbf{k}_n\rangle = \frac{\overset{+}{d}_{j_1\mathbf{k}_1} \ldots \overset{+}{d}_{j_n\mathbf{k}_n}}{\sqrt{(\nu_1! \ldots \nu_\alpha!)}} |0\rangle$$

These states are also eigenstates of $H_0{}^\delta$:

$$H_0{}^\delta |n\rangle = (\omega^\delta_{j_1\mathbf{k}_1} + \cdots + \omega^\delta_{j_n\mathbf{k}_n})|n\rangle$$

All these states generate a complete system of eigenstates in the vector space $\mathscr{H}^\delta$, i.e.

$$\otimes^\delta \underset{\mathrm{Df}}{=} |0\rangle\langle 0| + \sum_{n=1}^{\infty} \sum_{j_1 \ldots j_n = 0}^{\infty} (-)^{j_1 + \cdots + j_n} \int d\mathbf{k}_1 \ldots \int d\mathbf{k}_n |n\rangle\langle n| = 1$$

What does happen with the space $\mathscr{H}^\delta$ when $\delta \to 0$? At $\delta \to 0$ the masses of all ghosts increase according to (4.16). Therefore if any physical state is characterised by a definite value of energy then in the limit $\delta \to 0$ no physical states with arbitrary but finite energy can consist of ghost quanta. In this sense we have

$$\lim_{\delta \to 0} \mathscr{H}^\delta = \mathscr{H} \tag{4.18}$$

where $\mathscr{H}$ is the Hilbert space which contains

(1) a vacuum state $|0\rangle$, $a_\mathbf{k}|0\rangle = 0$,

(2) single- and many-particle states

$$|n\rangle = |\mathbf{k}_1, \ldots, \mathbf{k}_n\rangle = \frac{1}{\sqrt{(n!)}} \overset{+}{a}_{\mathbf{k}_1} \ldots \overset{+}{a}_{\mathbf{k}_n} |0\rangle$$

All these states generate the complete system in $\mathscr{H}$:

$$\otimes \underset{\mathrm{Df}}{=} |0\rangle\langle 0| + \sum_{n=1}^{\infty} \int d\mathbf{k}_1 \ldots \int d\mathbf{k}_n |\mathbf{k}_1, \ldots, \mathbf{k}_n\rangle\langle \mathbf{k}_1, \ldots, \mathbf{k}_n| = 1$$

5. *Green Functions of the Field $\phi^\delta(x)$*

First of all, let us consider the commutator

$$\Delta^\delta(x - y) = [\phi^\delta(x), \phi^\delta(y)]_- \tag{5.1}$$

Substituting the representation (4.8) into (5.1) and using (4.9) and (4.13) we obtain

$$\Delta^\delta(x) = \sum_{j=0}^{\infty} (-)^j A_j^\delta \Delta_j^\delta(x) \tag{5.2}$$

where

$$\Delta_j^\delta(x) = \frac{1}{(2\pi)^3} \int d^4k\epsilon(k_0)\delta(k^2 - m_j^2(\delta))\, e^{-ikx}$$
$$= \frac{1}{2\pi i}\epsilon(x_0)\delta(x^2) - \frac{m_j(\delta)}{4\pi i\sqrt{(x^2)}}\theta(x^2)J_1(m_j(\delta)\sqrt{(x^2)}) \qquad (5.3)$$

Because the series (5.2) converges absolutely we have

$$\Delta^\delta(x) = 0 \qquad \text{when } x^2 < 0 \qquad (5.4)$$

Thus the operator $\phi^\delta(x)$ satisfies the local commutation relations.

Now let us introduce $\Delta^\delta_{(\pm)}(x)$ functions according to

$$\Delta^\delta_{(-)}(x-y) = \Delta^\delta_{(+)}(y-x) = \langle 0|\phi^\delta(x)\phi^\delta(y)|0\rangle \qquad (5.5)$$

We have

$$\Delta^\delta_{(-)}(x) = \sum_{j=0}^{\infty} (-)^j A_j^\delta \Delta^\delta_{j(-)}(x) \qquad (5.6)$$

where

$$\Delta^\delta_{j(-)}(x) = \frac{1}{(2\pi)^3}\int d^4k\theta(k_0)\delta(k^2 - m_j^2(\delta))\, e^{-ikx} \qquad (5.7)$$

For $x^2 \to 0$, according to Bogolubov & Shirkov (1969), one can get

$$\Delta^\delta_{j(-)}(x) = -\frac{i}{4\pi}\epsilon(x_0)\delta(x^2) - \frac{1}{4\pi^2 x^2}$$
$$+ \frac{m_j^2(\delta)}{16\pi^2}\log\frac{m_j^2(\delta)|x^2|}{4} + \frac{im_j^2(\delta)}{16\pi}\epsilon(x_0)\theta(x^2) + O(x^2\log x^2)$$

Substituting this expansion into (5.6) we get

$$\Delta^\delta_{(-)}(x) = \frac{1}{16\pi^2}\sum_{j=0}^{\infty}(-)^j A_j^\delta m_j^2(\delta)\log m_j^2(\delta) + O(x^2\log x^2)$$

Here we have used the correlations (3.9). Hence it appears that the function $\Delta^\delta_{(-)}(x)$ is finite at $x = 0$ and

$$\Delta^\delta_{(-)}(0) = \frac{m^2}{16\pi^2}\sum_{j=0}^{\infty}(-)^j A_j^\delta \mu_j \log\mu_j < \infty \qquad (5.8)$$

It means that the operator $\phi^\delta(x)$ is well-defined because of

$$\langle 0|\phi^\delta(x)\phi^\delta(x)|0\rangle = \Delta^\delta_{(-)}(0) < \infty \qquad (5.9)$$

Let us consider the causal Green function

$$\Delta_c^\delta(x-y) = \langle 0|T(\phi^\delta(x)\phi^\delta(y))|0\rangle = \sum_{j=0}^{\infty} (-)^j A_j^\delta \Delta_{jc}^\delta(x-y) \qquad (5.10)$$

where

$$\Delta_{jc}^\delta(x) = \frac{1}{(2\pi)^4 i} \int \frac{dk\, \mathrm{e}^{-ikx}}{m_j^2(\delta) - k^2 - i\epsilon}$$

Otherwise

$$\Delta_c^\delta(x) = \frac{1}{(2\pi)^4 i} \int dk \tilde{\Delta}_c^\delta(k^2)\, \mathrm{e}^{-ikx} \qquad (5.11)$$

where

$$\begin{aligned} \tilde{\Delta}_c^\delta(k^2) &= \sum_{j=0}^{\infty} \frac{(-)^j A_j^\delta}{m_j^2(\delta) - k^2 - i\epsilon} \\ &= \frac{1}{m^2 - k^2 - i\epsilon} \sum_{n=0}^{\infty} \frac{v_n l^{2n}(k^2 - m^2)^n}{\prod_{j=1}^{n+2}\left(1 - \frac{\delta}{j}\frac{k^2 - m^2 + i\epsilon}{m^2}\right)} \end{aligned} \qquad (5.12)$$

The function $\tilde{\Delta}_c^\delta(k^2)$ is analytic in the complex k^2-plane for Im $k^2 \geqslant 0$ and

$$\tilde{\Delta}_c^\delta(k^2) = O\left(\frac{1}{|k^2|^3}\right)$$

when $k^2 \to \infty$ for Im $k^2 \geqslant 0$.

The retarded and advanced Green functions can be defined in the following way

$$\begin{aligned} \Delta_{\text{ret}}^\delta(x) &= \theta(x_0)\Delta^\delta(x) = \Delta_c^\delta(x) + \Delta_{(+)}^\delta(x) \\ \Delta_{\text{adv}}^\delta(x) &= -\theta(-x_0)\Delta^\delta(x) = \Delta_c^\delta(x) - \Delta_{(-)}^\delta(x) \end{aligned} \qquad (5.13)$$

They satisfy the conditions

$$\begin{aligned} \Delta_{\text{ret}}^\delta(x) &= 0 \qquad \text{for} \begin{cases} x^2 < 0 \\ x^2 > 0, x_0 < 0 \end{cases} \\ \Delta_{\text{adv}}^\delta(x) &= 0 \qquad \text{for} \begin{cases} x^2 < 0 \\ x^2 > 0, x_0 > 0 \end{cases} \end{aligned} \qquad (5.14)$$

Thus we can see that all Green functions satisfy all requirements of the local quantum field theory. It means that the field $\phi^\delta(x)$ is local.

The following correlations are valid for the Green functions $\Delta_c^\delta(x)$ and $\overset{\delta}{\Delta}_{(\pm)}(x)$:

$$\begin{aligned}\Delta_c^\delta(x) &= \theta(x_0)\overset{\delta}{\Delta}_{(-)}(x) + \theta(-x_0)\overset{\delta}{\Delta}_{(+)}(x)\\ \overset{\delta}{\Delta}_{(-)}(x) &= \theta(x_0)\Delta_c^\delta(x) + \theta(-x_0)\Delta_c^{\delta *}(x)\end{aligned} \tag{5.15}$$

An additional correlation should be mentioned because it is important when proving the unitarity of the S-matrix regularised:

$$\left\langle 0 \left| T\left(\frac{\partial}{\partial x_\mu}\phi^\delta(x)\frac{\partial}{\partial y_\nu}\phi^\delta(y)\right)\right| 0\right\rangle = \frac{\partial^2}{\partial x_\mu \partial y_\nu}\langle 0|T(\phi^\delta(x)\phi^\delta(y))|0\rangle \tag{5.16}$$

In other words, the T-product in the Wick sense coincides with the T-product in the Dyson sense, i.e. symbolically

$$T_{\mathrm{W}} = T_{\mathrm{D}} \tag{5.17}$$

Indeed, it is easy to obtain for the fields $\phi_j^\delta(x)$

$$\begin{aligned}&\frac{\partial^2}{\partial x_\mu \partial y_\nu}\langle 0|T(\phi_j^\delta(x)\phi_{j'}^\delta(y))|0\rangle\\ &\qquad = \left\langle 0 \left| T\left(\frac{\partial}{\partial x_\mu}\phi_j^\delta(x)\frac{\partial}{\partial y_\nu}\phi_{j'}^\delta(y)\right)\right| 0\right\rangle + i(-)^j\delta_{jj'}\delta_{\mu 0}\delta_{\nu 0}\delta^{(4)}(x-y)\end{aligned}$$

Therefore

$$\begin{aligned}&\frac{\partial^2}{\partial x_\mu \partial y_\nu}\langle 0|T(\phi^\delta(x)\phi^\delta(y)|0\rangle - \left\langle 0 \left| T\left(\frac{\partial}{\partial x_\mu}\phi^\delta(x)\frac{\partial}{\partial x_\nu}\phi^\delta(x)\right)\right| 0\right\rangle\\ &\qquad = i\delta_{\mu 0}\delta_{\nu 0}\delta^{(4)}(x-y)\sum_{j=0}^{\infty}(-)^j A_j^\delta = 0\end{aligned}$$

according to (3.9). Thus the following correlations take place

$$\begin{aligned}\partial_\mu\partial_\nu\Delta_c^\delta(x) &= \theta(x_0)\partial_\mu\partial_\nu\overset{\delta}{\Delta}_{(-)}(x) + \theta(-x_0)\partial_\mu\partial_\nu\overset{\delta}{\Delta}_{(+)}(x)\\ \partial_\mu\partial_\nu\overset{\delta}{\Delta}_{(-)}(x) &= \theta(x_0)\partial_\mu\partial_\nu\Delta_c^\delta(x) + \theta(-x_0)\partial_\mu\partial_\nu\Delta_c^{\delta *}(x)\end{aligned} \tag{5.18}$$

6. *The Green Functions in the Limit* $\delta \to 0$

The Green functions in the limit $\delta \to 0$ are distributions which are defined on a space of test functions Z_a. Therefore we have to consider improper transitions to the limit, i.e. investigate the limits

$$\lim_{\delta\to 0}\int dx G^\delta(x)f(x) = \lim_{\delta\to 0}\int dk\tilde{G}^\delta(k)\tilde{f}(k) = ? \tag{6.1}$$

where $G^\delta(x)$ is a Green function and $f(x) \in Z_a$.

Let us define our space of test functions Z_a. We say that $f(x_1, \ldots, x_n) \in Z_a$

if

(1) $f(z_1, \ldots, z_n)$ is an entire analytical function of n complex variables $z_\nu = x_\nu + iy_\nu$ ($\nu = 1, 2, \ldots, n$) for which there exist such $C > 0$ and $A_\nu > 0$ ($\nu = 1, \ldots, n$) that

$$|f(z_1, \ldots, z_n)| \leqslant C \exp\left\{ \sum_{\nu=1}^{n} A_\nu |z_\nu|^a \right\}$$

(2) for any y_ν ($\nu = 1, 2, \ldots, n$)

$$\int_{-\infty}^{\infty} dx_1 \ldots \int_{-\infty}^{\infty} dx_n |f(x_1 + iy_1, \ldots, x_n + iy_n)| < \infty$$

The number a is chosen depending on the interaction Lagrangian under consideration and the order of the form factor $K(k^2 l^2)$. It is necessary to choose

$$1 < a < \frac{2\rho}{2\rho - 1} \qquad \text{or} \qquad \rho < \frac{a}{2(a-1)} \tag{6.2}$$

The space $\tilde{Z}_a$, which is the space of Fourier transformations of functions $f \in Z_a$, consists of differentiable functions $\tilde{f}(p_1, \ldots, p_n)$ satisfying the condition

(3) there exist positive numbers $C > 0$ and $B_\nu > 0$ ($\nu = 1, \ldots, n$) that

$$|\tilde{f}(p_1, \ldots, p_n)| \leqslant C \exp\left\{ -\sum_{\nu=1}^{n} B_\nu |p_\nu|^\gamma \right\} \tag{6.3}$$

where

$$\gamma = \frac{a}{a-1} \qquad \text{and} \qquad \rho < \frac{\gamma}{2}$$

First of all, let us consider the commutator $\Delta^\delta(x)$. We have

$$\lim_{\delta \to 0} \int dx \Delta^\delta(x) f(x) = \lim_{\delta \to 0} \int dk \tilde{\Delta}^\delta(k) \tilde{f}(k)$$

$$= \int \frac{dk}{(2\pi)^3} \epsilon(k_0) \delta(k^2 - m^2) \tilde{f}(k) + \lim_{\delta \to 0} Q^\delta \tag{6.4}$$

where

$$Q^\delta = \int \frac{dk}{(2\pi)^3} \tilde{f}(k) \sum_{j=1}^{\infty} (-)^j A_j^\delta \epsilon(k_0) \delta(k^2 - m_j^2(\delta))$$

$$= \sum_{j=1}^{\infty} (-)^j A_j^\delta \int\int \frac{d\mathbf{k} dk_0}{(2\pi)^3} \tilde{f}(k_0, \mathbf{k}) \epsilon(k_0) \delta(k^2 - m_j^2(\delta))$$

The following estimation can be obtained

$$I_j^\delta = \left| \int\int \frac{d\mathbf{k}dk_0}{(2\pi)^3} \tilde{f}(k_0, \mathbf{k})\epsilon(k_0)\delta(k^2 - m_j^2(\sigma)) \right|$$

$$\leqslant \frac{1}{(2\pi)^3} \int \frac{d\mathbf{k}}{2\omega_{j\mathbf{k}}^\delta} [|\tilde{f}(\omega_{j\mathbf{k}}^\delta, \mathbf{k})| + |\tilde{f}(-\omega_{j\mathbf{k}}^\delta, \mathbf{k})|]$$

$$\leqslant \text{const.} \int \frac{d\mathbf{k}}{\omega_{j\mathbf{k}}^\delta} \exp\{-B[(\omega_{j\mathbf{k}}^\delta)^\gamma + |\mathbf{k}|^\gamma]\}$$

Making use of the inequality

$$(\omega_{j\mathbf{k}}^\delta)^\gamma + |\mathbf{k}|^\gamma = (\sqrt{(\mathbf{k}^2 + m_j^2(\delta))})^\gamma + |\mathbf{k}|^\gamma \geqslant h_1(m_j(\delta))^\gamma + h_2|\mathbf{k}|^\gamma$$

where $h_1 = 2^{(\gamma/2)-1}$, $h_2 = 1 + 2^{(\gamma/2)-1}$, now Q^α is evaluated to be

$$|Q^\alpha| \leqslant \sum_{j=1}^\infty A_j^\delta I_j^\delta \leqslant \text{const.} \sum_{j=1}^\infty A_j^\delta \exp(-Bh_1(m_j(\delta))^\gamma) \int \frac{d\mathbf{k} \exp(-Bh_2|\mathbf{k}|^\gamma)}{\sqrt{[m_j^2(\delta) + \mathbf{k}^2]}}$$

$$\leqslant \text{const.} \sum_{j=1}^\infty \frac{(1+N)^2}{N^j} \cdot \frac{\delta}{j} \exp\left\{ b\left[(1+N)\frac{j}{\delta}\right]^\rho - Bh_1\left(m\sqrt{1+\frac{j}{\delta}}\right)^\gamma \right\}$$

Because of $\rho < \gamma/2$, we have

$$|Q^\delta| \leqslant \delta \text{ . const.} \sum_{j=1}^\infty \frac{(1+N)^2}{N^j j} \leqslant \delta \text{ . const.}$$

and finally

$$\lim_{\delta \to 0} |Q^\delta| = 0$$

Thus in the limit $\delta \to 0$ the commutator $\Delta^\delta(x)$ changes into the commutator of the scalar field $\varphi(x)$

$$\lim_{\delta \to 0} \Delta^\delta(x) = \Delta(x) = \frac{1}{(2\pi)^3} \int dk\epsilon(k_0)\delta(k^2 - m^2)\, e^{-ikx} \tag{6.5}$$

In the same manner it is easy to show that in the improper sense

$$\lim_{\delta \to 0} \Delta_{(\pm)}^\delta(x) = \Delta_{(\pm)}(x) = \frac{1}{(2\pi)^3} \int dk\theta(\mp k_0)\delta(k^2 - m^2)\, e^{-ikx} \tag{6.6}$$

The existence of these limits means that there exists a weak limit

$$\lim_{\delta \to 0} \phi^\delta(x) = \varphi(x) \tag{6.7}$$

where $\varphi(x)$ is the scalar field of mass m satisfying the free wave equation

$$(\Box - m^2)\varphi(x) = 0$$

Consequently, all ghost states disappear in the limit $\delta \to 0$. This result confirms the statement that

$$\lim_{\delta \to 0} \mathscr{H}^{\delta} = \mathscr{H}$$

which was accomplished in Section 4.

Consider now the causal function $\Delta_c^{\delta}(x)$. We have

$$\lim_{\delta \to 0} \int dx \Delta_c^{\delta}(x) f(x) = \lim_{\delta \to 0} \int \frac{dk}{(2\pi)^4 i} \tilde{\Delta}_c^{\delta}(k^2) \tilde{f}(k)$$

$$= \lim_{\delta \to 0} \int \frac{dk}{(2\pi)^4 i} \frac{\tilde{f}(k)}{m^2 - k^2 - i\epsilon} \sum_{n=0}^{\infty} \frac{v_n l^{2n} (k^2 - m^2)^n}{\prod_{j=1}^{n+2} \left(1 - \frac{\delta}{j} \frac{k^2 - m^2 - i\epsilon}{m^2}\right)}$$

$$= \frac{1}{(2\pi)^4 i} \int \frac{dk V(k^2 l^2) \tilde{f}(k)}{m^2 - k^2 - i\epsilon}$$

This integral converges because the estimation

$$|V(k^2 l^2) \tilde{f}(k)| \leqslant C \exp\{b l^{2\rho} |k_0^2 - \mathbf{k}^2|^{\rho} - B[|k_0|^{\gamma} + |\mathbf{k}|^{\gamma}]\}$$

is valid, and $\rho < \gamma/2$.

Thus the causal function $\Delta_c^{\delta}(x)$ changes in the limit $\delta \to 0$ into a non-local propagator

$$\lim_{\delta \to 0} \Delta_c^{\delta}(k) = \frac{V(k^2 l^2)}{m^2 - k^2 - i\epsilon} = \tilde{D}_c(k^2) \tag{6.8}$$

The function $\tilde{D}_c(k^2)$ has a single pole at $k^2 = m^2$. This pole corresponds to the scalar particle with mass m. The poles corresponding to all ghost states have disappeared. The form factor $V(k^2 l^2)$ is an entire function and it does not correspond to any real state. This form factor describes a non-local character of the interaction of our scalar particles.

Thus in the limit $\delta \to 0$ our theory becomes non-local.

Consequently, the non-local character of the interaction of the classical field (2.1) is revealed in quantum field theory as a residual effect of non-physical ghost states when these ghosts are removed by the transition to the limit $\delta \to 0$.

7. *Unitarity of the S-Matrix*

The interacting system is described by the Lagrangian density (4.7). The total Hamiltonian of this system has the form

$$H^{\delta} = H_0^{\delta} + H_I^{\delta} \tag{7.1}$$

Here $H_0{}^\delta$ is given by (4.15) and

$$H_I^\delta = -g \int d\mathrm{x} : U(\phi^\delta(\mathrm{x}, 0)): \tag{7.2}$$

where $\phi^\delta(\mathrm{x}, 0)$ is defined by (4.6).

Let us find the S-matrix corresponding to the interaction Hamiltonian (7.2). According to the standard procedure, a counter-term should be introduced to provide the construction of the finite S-matrix on the mass shell. These counter-terms are responsible for the renormalisation of the vacuum energy, the mass and the wave function of our scalar particle. In order to take into account correctly the contributions of the interaction Hamiltonian and the counter-terms it is necessary to introduce an operation of 'switching on' and 'switching off' the interaction. Thus the regularised S-matrix can be written in the form

$$\begin{aligned} S^{\delta,L} = T \exp\Bigg\{ i \int dx \Bigg[& g\left(\frac{x}{L}\right) : U(\phi^\delta(x)) : \\ & - \tfrac{1}{2}\delta m^2\left(g\left(\frac{x}{L}\right)\right) : \phi^\delta(x)\phi^\delta(x) : \\ & - \tfrac{1}{2} Z_2\left(g\left(\frac{x}{L}\right)\right) : \phi^\delta(x)(\Box - m^2)\phi^\delta(x) : - E\left(g\left(\frac{x}{L}\right)\right)\Bigg]\Bigg\} \end{aligned} \tag{7.3}$$

Here

$$\begin{aligned} \delta m^2(g(x)) &= \sum_{n=2}^{\infty} \delta m^2_{(n)} [g(x)]^n \\ Z_2(g(x)) &= \sum_{n=2}^{\infty} Z_{2(n)} [g(x)]^n \\ E(g(x)) &= \sum_{n=2}^{\infty} E_{(n)} [g(x)]^n \end{aligned} \tag{7.4}$$

The counter-terms $\delta m^2(g)$ and $Z_2(g)$ are responsible for the renormalisation of the mass operator of scalar particle $\Sigma(p^2)$, so that in the limits $\delta \to 0$ and $L \to \infty$

$$\Sigma_r(p^2) = \Sigma(p^2) - \delta m^2 - Z_2(p^2 - m^2)$$

and

$$\Sigma_r(m^2) = \Sigma_r'(m^2) = 0$$

in each perturbation order. The counter-term $E(g)$ is responsible for removing the amplitude of the vacuum–vacuum transition.

The large parameter L defines the intensity of switching on the interaction. The function $g(x)$ satisfies the conditions

(1) $0 \leqslant g(x) \leqslant g$

(2) $g(0) = g$

(3) $\partial^s g(x)|_{x=0} = 0 \qquad$ for $s = 1, 2, 3, 4$

where $\partial^s = \partial_{\mu_1} \ldots \partial_{\mu_s}$

(4) $\int g(x) dx < \infty$

(5) $g(x) \in Z_a$

Our problem is to investigate the perturbation series for the $S^{\delta,L}$-matrix

$$S^{\delta,L} = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \int dx_1 \ldots \int dx_n g\left(\frac{x_1}{L}\right) \ldots g\left(\frac{x_n}{L}\right) S_n{}^{\delta}(x_1, \ldots, x_n) \quad (7.5)$$

It is necessary to show that there exists the following sequence of limits in each perturbation order

$$S^L = \lim_{\delta \to 0} S^{\delta,L} \quad (7.6)$$

$$S = \lim_{L \to \infty} S^L \quad (7.7)$$

Further we have to prove that the S-matrix (7.7) is unitary in perturbation theory, i.e.

$$SS^+ = 1 \quad (7.8)$$

This proof is given in Alebastrov & Efimov (1972, 1973). It is completely correct in the case under consideration.

Acknowledgements

The author wishes to thank Professor D. I. Blokhintsev and Professor V. Ya. Faynberg for useful discussions.

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